UNIT 1

Distributions

- Discrete distribution
  - (i) Binomial distribution.
  - (ii) Poisson distribution.
  - (iii) Geometric distribution.
- Continuous distribution
  - (i) Uniform distribution
  - (ii) Exponential distribution
  - (iii) Gamma distribution
  - (iv) Normal distribution

Discrete Distributions

Binomial distribution

Defn:-

A random variable x is said to follow binomial distribution if it assumes only non-negative values and its probability mass function (pmf) is given by

$$P\{X=x\} = P(x) = {}^nC_x\, p^x\, q^{n-x}$$

where $x = 0, 1, 2, \ldots n$.

where n and p are parameters, such that $p+q=1$

Here p is the prob. of success & q is the prob. of failure.

(2)

Moment generating function (M.G.F.) of Binomial distribution.

By defn. of M.G.F.

$$M_X(t) = E(e^{tX})$$

Since X is discrete r.v.

$$= \sum e^{tx} P(x)$$

Since X follows Binomial distribution.

$$= \sum_{x=0}^{n} e^{tx} \, {}^nC_x p^x q^{n-x}.$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}.$$

$$= {}^nC_0 (pe^t)^0 q^n + {}^nC_1 (pe^t) q^{n-1} + \cdots + {}^nC_n (pe^t)^n$$

$$= q^n + {}^nC_1 (pe^t) q^{n-1} + \cdots + (pe^t)^n$$

$$\boxed{M_X(t) = [q + pe^t]^n}$$

which is the M.G.F. of Binomial distribution.

To find Mean & Variance of Binomial distribution

First we find $E(X)$ and $E(X^2)$

W.K.T.

$$E[X] = \left\{\frac{d}{dt}[M_X(t)]\right\}_{t=0}$$

$$= \left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0} = \left[n(q+pe^t)^{n-1} pe^t\right]_{t=0}$$

$$= n(q+pe^0)^{n-1} pe^0 = n(q+p)p$$

$$= n(1)p \quad \text{since } p+q=1$$

$$\Rightarrow \boxed{E(X) = np}$$

ie., $\boxed{\text{Mean} = np}$

Now

$$E[X^2] = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}\left\{\frac{d}{dt}(q+pe^t)^n\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{n(q+pe^t)^{n-1} pe^t\right\}\right]_{t=0}$$

$$= \left[np\,\frac{d}{dt}\left\{(q+pe^t)^{n-1} e^t\right\}\right]_{t=0}$$

$$= \left[np\left\{(q+pe^t)^{n-1} e^t + (n-1)(q+pe^t)^{n-2}(pe^t)e^t\right\}\right]_{t=0}$$

$$= np\left\{(q+pe^0)^{n-1} e^0 + (n-1)(q+pe^0)^{n-2}(pe^0)e^0\right\}$$

(4)

$$= np\{(q+p) + (n-1)(q+p)^{n-2}\}$$

$$= np\{1 + (n-1)p(1)\} \quad \text{since } p+q=1$$

$$= np + n(n-1)p^2$$

$$\Rightarrow \boxed{E(x^2) = n(n-1)p^2 + np}$$

Now $\quad var(x) = E\{x^2\} - \{E(x)\}^2$

$$= n(n-1)p^2 + np - \{np\}^2$$

$$= n(n^2-n)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2 = np(1-p)$$

$$\boxed{var(x) = npq} \quad \text{since } 1-p=q$$

∴ In Binomial distribution

$$\text{Mean} = E(x) = np$$

$$\text{Variance}(x) = npq$$

POISSON DISTRIBUTION

Statement

A random variable X is said to follow Poisson distribution, if it assumes only non-negative values.

Its Pmf is given by

$$P\{X=x\} = P\{x\} = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where $\lambda > 0$ and $x = 0, 1, 2, \ldots \infty$.

where λ is the parameter.

(8)

Show that Poisson distribution is the limiting case of Binomial distribution.

Proof:

We Prove the required fact by assuming the three restrictions.

(i) the no. of trials, $n \to \infty$

(ii) the Prob. of success p, which is very small, ie., $p \to 0$

(iii) Let $\lambda = np \Rightarrow \boxed{p = \lambda/n}$ and $\boxed{q = 1 - \frac{\lambda}{n}}$

W.K.T. The Pmf of ~~Poisson~~ Binomial distribution is

$$P\{X=x\} = {}^nC_x p^x q^{n-x} \quad \text{where } x = 0, 1, 2, \ldots n.$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n.(n-1)(n-2)\cdots(n-x-1)(n-x)!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n.n\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{x-1}{n}\right)}{x!} \left(\frac{\lambda^x}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!} \cdot \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

When $n \to \infty$.

$$= \frac{(1)(1)\cdots(1)}{x!}\lambda^x e^{-\lambda}(1)$$

as $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

and $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$

$$\boxed{P(x) = \frac{e^{-\lambda}\lambda^x}{x!}}$$

where $x = 0, 1, 2, \ldots \infty$.

which is the pmf of Poisson distribution.

M.G.F. of Poisson Distribution

By defn. of M.G.F.

$$M_x(t) = E(e^{tx})$$

since x is a discrete random variable

$$= \sum e^{tx} P(x)$$

since x follows Poisson distribution

$$= \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots \infty\right\} = e^{-\lambda}e^{\lambda e^t}$$

$$= e^{\lambda e^t - \lambda}$$

$$\boxed{M_x(t) = e^{\lambda(e^t-1)}}$$

To find Mean & variance of Poisson distribution.

First we find $E(x)$ and $E(x^2)$.

W.K.T. $E(x) = \left\{\frac{d}{dt}(M_x(t))\right\}_{t=0}$.

$$= \left[\frac{d}{dt}\left\{e^{\lambda(e^t-1)}\right\}\right]_{t=0} = \left[\frac{d}{dt}\left\{e^{\lambda e^t-\lambda}\right\}\right]_{t=0}.$$

$$= \left[e^{\lambda e^t-\lambda} \cdot \lambda e^t\right]_{t=0} \quad \cdots\cdots (*)$$

$$= e^{\lambda e^0-\lambda} \cdot \lambda e^0 = e^{\lambda-\lambda} \cdot \lambda(1) = e^0 \cdot \lambda$$

$\Rightarrow$ $\boxed{E(x) = \lambda}$ ie., $\boxed{\text{Mean} = \lambda}$

Now

$$E\{x^2(t)\} = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{t=0}.$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt}(M_x(t))\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt}\left(e^{\lambda(e^t-1)}\right)\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt}\left(e^{\lambda e^t-\lambda}\right)\right\}\right]_{t=0} = \left[\frac{d}{dt}\left(e^{\lambda e^t-\lambda}\ \lambda e^t\right)\right]_{t=0}$$

$$= \lambda \left[\frac{d}{dt} \left(e^{\lambda e^t - \lambda} \cdot e^t \right) \right]_{t=0}$$

$$= \lambda \left[e^{\lambda e^t - \lambda} e^t + e^t e^{\lambda e^t - \lambda} \lambda e^t \right]_{t=0}$$

$$= \lambda \left[e^{\lambda e^0 - \lambda} e^0 + e^0 e^{\lambda e^0 - \lambda} \lambda e^0 \right]$$

$$= \lambda \left[e^{\lambda - \lambda} (1) + (1) e^0 \cdot \lambda (1) \right]$$

$$= \lambda \left[e^0 + \lambda (1) \right] = \lambda \left[1 + \lambda \right] = \lambda + \lambda^2$$

$$\Rightarrow \boxed{E\{x^2\} = \lambda^2 + \lambda}$$

Also,

$$\text{Var}(X) = E(x^2) - \left(E(X)\right)^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Var}(X) = \lambda}$$

In Poisson distribution

$$\boxed{\begin{aligned} \text{Mean} &= \lambda \\ \text{Variance} &= \lambda \end{aligned}}$$

NOTE:- Poisson distribution is the only discrete distribution to which $\boxed{\text{Mean} = \text{variance} = \lambda}$

Exponential Distribution

Smt.:-

A continuous random r.v. X is said to follow exponential distribution if its pdf is given by

$$\boxed{f(x) = \lambda e^{-\lambda x}}, \quad \text{where } \lambda > 0 \text{ \& } x \geq 0.$$

To find the M.G.F. of Exponential distribution

By defn. of M.G.F.

$$M_x(t) = E(e^{tx})$$

Since X is a continuous r.v.

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Here X follows exponential distribution.

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$= \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{-\lambda x + tx}\, dx.$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty}$$

$$= -\frac{\lambda}{\lambda - t} \left[e^{-(\lambda-t)x} \right]_0^{\infty} = -\frac{\lambda}{\lambda - t} \left[e^{-\infty} - e^{0} \right]$$

$$= -\frac{\lambda}{\lambda - t} (0 - 1) = \frac{\lambda}{\lambda - t}.$$

$$\Rightarrow \boxed{M_x(t) = \frac{\lambda}{\lambda - t}}$$

which is the M.G.F. of exponential distribution.

To find the Mean & variance of exponential distribution

W.K.T.

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

$$= \frac{\lambda}{\lambda\left(1 - \frac{t}{\lambda}\right)} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$\Rightarrow M_x(t) = 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \cdots + \left(\frac{t}{\lambda}\right)^n + \cdots \infty.$$

$$\Rightarrow M_x(t) = 1 + \left(\frac{1}{\lambda}\right)t + \frac{1}{\lambda^2}t^2 + \frac{1}{\lambda^3}t^3 + \cdots + \frac{1}{\lambda^n}t^n + \cdots \infty$$

Now.

$$\mu_1' = E(X) = \text{Co-eff. of } \frac{t}{1!}$$

$$\Rightarrow E(X) = \frac{1}{\lambda}$$

In general,

$$\mu_r' = \text{Co-eff. of } \frac{t^r}{r!}$$

ie., $\boxed{\text{Mean} = \frac{1}{\lambda}}$

$$\mu_2' = E(X^2) = \text{Co-eff. of } \frac{t^2}{2!} = \frac{2}{\lambda^2}$$

ie., $E(X^2) = 2/\lambda^2$

Now $\text{var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

$$\Rightarrow \boxed{\text{var}(x) = \frac{1}{\lambda^2}}$$

Memoryless Property of Exponential distribution

Smt. :-

If the random variable X follows exponential distribution. then

$$P\left[X>(m+n)/X>m\right] = P[X>n] \quad \text{for all } m,n>0.$$

Proof

Given X follows exponential distribution.

$$f(x) = \lambda e^{-\lambda x} \quad \text{where } x \geq 0.$$

LHS

$$P\left[X>(m+n)/X>m\right] = \frac{P[(X>(m+n)) \cap (X>m)]}{P[X>m]}$$

$$\Rightarrow P\left[X>m+n/X>m\right] = \frac{P[X>m+n]}{P[X>m]} \quad \text{---- (*)}$$

Since

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

Now, for any +ve integer k,

$$P[X>k] = \int_k^\infty f(x)\,dx.$$

Since X is continuous r.v.

$$= \int_k^\infty \lambda e^{-\lambda x}\,dx = \lambda \int_k^\infty e^{-\lambda x}\,dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty$$

$$= -\left[e^{-\lambda x}\right]_k^\infty = -\left[e^{-\infty} - e^{-\lambda k}\right]$$

$$= -\left[0 - e^{-\lambda k}\right]$$

$$\Rightarrow \boxed{P[X>k] = e^{-\lambda k}}$$

using in (*)

$$P\left[X > m+n \,/\, X>m\right] = \frac{P[X>m+n]}{P[X>m]}$$

$$= \frac{e^{-\lambda(m+n)}}{e^{-\lambda m}} = \frac{e^{-\lambda m - \lambda n}}{e^{-\lambda m}}$$

$$= \frac{e^{-\lambda m} \cdot e^{-\lambda n}}{e^{-\lambda m}} = e^{-\lambda n}$$

$$= P(X>n)$$

$$\Rightarrow \boxed{P\left[X > m+n \,/\, X>m\right] = P(X>n)}$$

Smt:- **NORHAL DISTRIBUTION**

A continuous random variable x said to follows Normal distribution if its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z^2)}$$

where $z = \frac{x-\mu}{\sigma}$

$-\infty < x < \infty$.

To find the M.G.f. of Normal distribution

By defn. of M.G.f.

$$M_x(t) = E(e^{tx})$$

Here x is a continuous r.v.

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

Since x follows normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \text{ where } -\infty < x < \infty.$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow \sigma z = x - \mu$

$\Rightarrow \sigma dz = dx.$

$\Rightarrow z = \frac{x-\mu}{\sigma}$

When $x = -\infty \Rightarrow z = -\infty$

$x = \infty \Rightarrow z = \infty$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \, e^{-\frac{1}{2}(z^2)} \, (\sigma\, dz)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + \sigma t z + t\mu} \, dz .$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + \sigma t z} \, dz \quad (\cdot X \cdot) .$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[z^2 - 2\sigma t z\right]} \, dz .$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{(z-\sigma t)^2 - \sigma^2 t^2\right\}} \, dz .$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-\sigma t)^2/2} \cdot e^{\sigma^2 t^2/2} \, dz$$

$$= \frac{e^{\mu t} \cdot e^{\sigma^2 t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-\sigma t)^2/2} \, dz .$$

Put $z - \sigma t = u$

$dz = du$

when $z = -\infty \Rightarrow u = -\infty$

$z = \infty \Rightarrow u = \infty$

$$\therefore M_x(t) = \frac{e^{t\mu}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} e^{-u^2/2}\, du$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} e^{\frac{\sigma^2 t^2}{2}} \sqrt{2\pi} \quad \left[\text{Since } \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \text{ Gamma integral}\right]$$

$$= e^{\mu t} \cdot e^{\sigma^2 t^2/2}$$

$$= e^{(\mu t + \sigma^2 t^2/2)}$$

$$\therefore M_x(t) = e^{(\mu t + \sigma^2 t^2/2)}$$

To find mean and variance of normal distribution

Mean

$$\text{Mean} = E(x) = M_x'(0)$$

$$M_x(t) = e^{(\mu t + \sigma^2 t^2/2)}$$

$$M_x'(t) = e^{\mu t + \sigma^2 t^2/2} \left(\mu + \frac{2\sigma^2 t}{2}\right)$$

$$M_x'(0) = e^0 (\mu) = \mu$$

$$\therefore \boxed{E(x) = \mu}$$

Variance

$$V(x) = E(x^2) - (E(x))^2$$

$$E(x^2) = M_x''(0)$$

$$M_x''(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\sigma^2) + (\mu + \sigma^2 t)\, e^{\mu t + \sigma^2 t^2/2} \left(\mu + \frac{2\sigma^2 t}{2}\right)$$

$$= e^{\mu t + \sigma^2 t^2/2} \left[\sigma^2 + (\mu + \sigma^2 t)^2\right]$$

$$M_x''(0) = e^0 \left[(\sigma^2) + \mu^2\right]$$

$\therefore \quad M_x''(0) = \sigma^2 + \mu^2 \qquad = E(x^2)$

$\text{Var}(x) = E(x^2) - (E(x))^2$

$= \sigma^2 + \mu^2 - (\mu)^2$

$= \sigma^2 + \mu^2 - \mu^2$

$= \sigma^2$

$\therefore \quad \text{Var}(x) = \sigma^2$

$$\text{Mean} = \mu$$

$$\text{Var}(x) = \sigma^2$$